

# DECAY OF CORRELATIONS, QUANTITATIVE RECURRENCE AND LOGARITHM LAW FOR ROVELLA ATTRACTORS.

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ABSTRACT. In this paper we prove that a class of skew products maps with non uniformly hyperbolic base has exponential decay of correlations. We apply this to obtain a logarithm law for the hitting time associated to a *contracting Lorenz (or Rovella)* attractor at all the points having a well defined local dimension, and a quantitative recurrence estimation.

## 1. INTRODUCTION

The term *statistical properties* of a dynamical system  $F : X \rightarrow X$ , where  $X$  is a measurable space and  $F$  a measurable map, refers to the long time behavior of large sets of trajectories of the system. It is well known that this relates to the properties of transfer operator, a linear operator associated to the dynamics that embodies how the measures evolve under the action of the system.

Statistical properties are often a better object to be studied than pointwise behavior. In fact, the future behavior of initial data can be unpredictable, but statistical properties are often regular and their description simpler. Suitable results can be established in many cases, to relate the evolution of measures with that of large sets of points (ergodic theorems, large deviations, central limit, logarithm law, etc...).

In this paper we take the point of view of studying the evolution of measures and its speed of convergence to equilibrium to understand the statistical properties of a class of dynamical systems.

We consider the *contracting Geometric Lorenz flows* and its perturbations [28], a relative of the Lorenz flow, but strictly non-uniformly hyperbolic; following the cited paper, we have that if the Geometric Flow admits an attractor and we take a one parameter family of perturbations, parametrized by a parameter  $a$ , there exists an  $E \subset [0, a_0)$  of parameters called the **Rovella parameters**, of positive Lebesgue measure, that admit an attractor.

We prove that the decay of correlations for the map  $F_a$ , induced on a Poincaré section for a Rovella parameter  $a \in E$  is exponential with respect

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to Lipschitz observables: this permits us to study some of the statistical features of the map and the flow.

We follow the line of [2] exploiting the fact that the system has a preserved contracting fibration and it can be seen as a skew product whose base transformation has exponential convergence to equilibrium (a measure of how fast iterates of the Lebesgue measure converge to the physical measure), proving the following.

**Theorem A.** *For all Rovella parameters  $a \in E$ , the two dimensional map  $F_a$  associated to the flow has exponential decay of correlations with respect to Lipschitz observables : there exists  $C, \Lambda \in \mathbb{R}^+$ ,  $\Lambda < 1$  such that*

$$\left| \int f \cdot (g \circ F_a^n) d\mu - \int g d\mu \int f d\mu \right| \leq C \Lambda^n \cdot \|g\|_{Lip} \cdot \|f\|_{Lip}.$$

The rapid decay of correlations give, as a consequence, a quantitative recurrence estimation and an estimation for the scaling behavior of the time which is needed to hit small targets (Section 7).

Denote by  $\tau_r(x, x_0)$  the time needed for the  $F_a$  orbit (respectively the  $X$ -orbit) of a point  $x$  to enter the ball  $B_r(x_0)$ , and by  $\tau_r(x_0)$ , the time needed for the  $F_a$  orbit (respectively the  $X$ -orbit) of a point  $x_0$  to return in the ball  $B_r(x_0)$ .

**Theorem B.** *For the Rovella map  $F_a$ ,  $a \in E$ , where  $E$  is the set of Rovella parameters, and  $\mu_{F_a}$  is the invariant SBR measure for the map  $F_a$ , then for each regular point  $x_0 \in \Sigma_a$  such that  $d_{\mu_{F_a}}(x_0)$  exists, it holds*

$$\lim_{r \rightarrow 0} \frac{\log \tau_r^{F_a}(x, x_0)}{-\log r} = d_{\mu_{F_a}}(x_0)$$

for  $\mu_{F_a}$ -almost each  $x \in \Sigma_a$ .

For  $\mu_{F_a}$  almost every  $x$

$$\liminf_{r \rightarrow 0} \frac{\log \tau_r^{F_a}(x_0)}{-\log r} = \underline{d}_{\mu_{F_a}}(x_0), \quad \limsup_{r \rightarrow 0} \frac{\log \tau_r^{F_a}(x_0)}{-\log r} = \bar{d}_{\mu_{F_a}}(x_0).$$

The logarithm law for the two dimensional map and the integrability of the return time with respect to  $\mu_{F_a}$  imply a logarithm laws for the hitting time and recurrence time of the flow.

**Theorem C.** *If  $X_a^t$  is a Rovella flow with  $a \in E$ , where  $E$  is the set of Rovella parameters, and  $\mu_{X_a}$  is an invariant SBR measure for the flow  $X_a$ , then for each regular point  $x_0 \in \mathbb{R}^3$  such that  $d_{\mu_{X_a}}(x_0)$  exists, it holds*

$$\lim_{r \rightarrow 0} \frac{\log \tau_r^{X_a^t}(x, x_0)}{-\log r} = d_{\mu_{X_a}}(x_0) - 1$$

for  $\mu_{X_a}$ -almost each  $x \in \mathbb{R}^3$ . While for  $\mu_{X_a}$  almost every  $x$

$$\liminf_{r \rightarrow 0} \frac{\log \tau_r^{X_a}(x)}{-\log r} = \underline{d}_{\mu_{X_a}}(x) - 1, \quad \limsup_{r \rightarrow 0} \frac{\log \tau_r^{X_a}(x)}{-\log r} = \bar{d}_{\mu_{X_a}}(x) - 1.$$

This extends the results of [8] to the Geometric Rovella flow and its perturbations.

**1.1. Organization of the text.** This paper is organized as follows. In Section 2 we explain the main object of this article, the Rovella flow. In Section 3 we explicit the main properties of the one dimensional map associated to a Rovella flow. In Section 4 we recall some general results on convergence and correlation decay for skew-products with contracting fibers from [2], that will be used to proof Theorem A. In Section 5 we show how to extend some results about decay of correlations and convergence to equilibrium for Hölder observables to generalized bounded variation observables. This extension is necessary to apply the general results of the previous section to our case. In Section 6 we establish exponential decay of correlations with respect to Lipschitz observables for the two dimensional Rovella map and proof of Theorem A. In Section 7 we show some consequences of the decay of correlations proved above as hitting time and quantitative recurrence estimations. Finally, in Section 8 we explain a result about linearization and properties of the Poincaré map associated to a Rovella flow needed along the paper.

## 2. THE ROVELLA FLOW

In this section we present the family of dynamical systems which are the object of our paper.

The Rovella family was introduced in [28]. The starting point for its definition is the Geometric Rovella Flow, a strict relative of the Geometric Lorenz system, in which the uniformly expanding direction is replaced by a strict nonuniformly expanding direction.

We describe informally the construction of the Geometric Rovella flow, following [26]. Let  $(\dot{x}, \dot{y}, \dot{z}) = (\lambda_1 x, \lambda_2 y, \lambda_3 z)$  be a vector field in the cube  $[-1, 1]^3$ , with a singularity at the origin  $(0, 0, 0)$ . Suppose the eigenvalues  $\lambda_i$ ,  $1 \leq i \leq 3$  satisfy the relations

$$(2.1) \quad -\lambda_2 > -\lambda_3 > \lambda_1 > 0, \quad r = -\frac{\lambda_2}{\lambda_1}, \quad s = -\frac{\lambda_3}{\lambda_1}, \quad r > s + 3.$$

It is worth to remark that  $\lambda_1 + \lambda_3 < 0$  while in the definition of the Geometric Lorenz the condition is  $\lambda_1 + \lambda_3 > 0$ . The condition  $r > s + 3$  is used in the original paper [28] to guarantee the existence of a  $C^3$  uniformly contracting stable foliation for the Poincaré first return map of the Geometric Rovella Flow and its perturbations.

Let  $\Sigma^- = \{(x, y, 1) \mid -1/2 \leq x \leq 0, |y| \leq 1/2\}$ ,  $\Sigma^+ = \{(x, y, 1) \mid 0 \leq x \leq 1/2, |y| \leq 1/2\}$  and  $\Sigma = \Sigma^+ \cup \Sigma^-$ . In figure 1 we can see the behaviour of the field near the origin, with some computations, it is possible to see that the flow reaches the transverse section  $x = 1$  (a similar reasoning works for  $x = -1$ ) obeying the following law:

$$(2.2) \quad \tilde{F}(x, y, 1) = (1, yx^r, x^s)$$

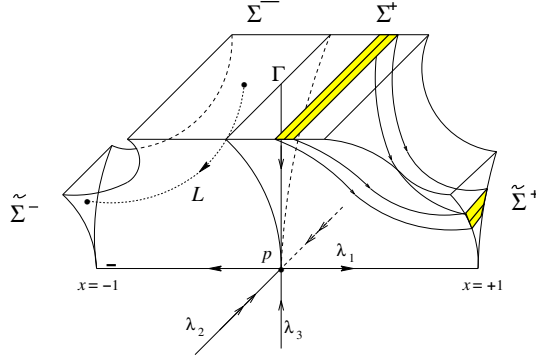


FIGURE 1. Behaviour near the origin.

Outside the cube, as in the Geometric Lorenz case, to imitate the random turns of a regular orbit around the origin and obtain a butterfly shape for our flow, we let the flow return to the cross section  $\Sigma$  through a flow described by a suitable composition of a rotation  $R_{\pm}$ , an expansion  $E_{\pm\rho}$  and a translation  $T_{\pm}$ . The resulting effect of the flow outside the cube when we arrive on  $\Sigma$  may be represented by a rotation and the expansion which have the form:

$$R_{\pm}(x, y, z) = \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & 1 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix}, \quad E_{\pm\rho}(x, y, z) = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\rho$  is such that  $\rho \cdot (1/2)^s < 1$ . This condition insures that the image of the map is contained in  $\Sigma$ .

The translations  $T_{\pm}$  are chosen in such a way that the unstable direction starting from the origin is sent to the boundary of  $\Sigma$  and the images of  $\tilde{\Sigma}_{\pm}$  are disjoint.

It is possible to find a flow that realizes this construction, as it is described in [8, 26].

Composing the expression in (2.2) with  $R_{\pm}, E_{\pm\rho}$  and  $T_{\pm}$  we write an explicit formula for  $F_0 : \Sigma \rightarrow \Sigma$ , the Poincaré first return maps of the Geometric Rovella Flow on the section  $\Sigma$ :

$$(2.3) \quad F_0(x, y) = (T_0(x), G_0(x, y))$$

$$T_0(x) = \begin{cases} -\rho|x|^s + 1/2 & x > 0 \\ \rho|x|^s - 1/2 & x < 0 \end{cases}, \quad G_0(x, y) = \begin{cases} y|x|^r + c_0 & x > 0 \\ -y|x|^r + c_1 & x < 0 \end{cases},$$

where  $c_0, c_1$  are real numbers depending on the choice of the translations  $T_{\pm}$ .

Next, let us recall the definition of an attractor.

**Definition 1.** Let  $X$  be a vector field, with associated flow  $X^t$ . A set  $\Lambda$  is an attractor for  $X$  if it is compact, invariant under  $X^t$ , transitive (i.e., contains a dense orbit) and it has a compact neighbourhood  $U$  such that  $\Lambda = \bigcap_{t \geq 0} X^t(U)$ .

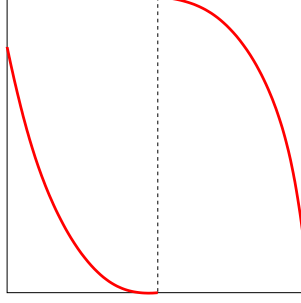


FIGURE 2. A 1-dimensional Rovella map

Note that the transitivity and the existence of an a.c.i.m. for  $T_0$  depends on the constant  $\rho$ .

**Remark 2.1.** If  $\rho = (1/2)^{-s}$ , then the map  $T_0$  is topologically conjugated to the doubling map and it is called, in the literature, a “full Rovella map” [26]. For a “full Rovella map”, the existence of an attractor and of an a.c.i.m. for  $T_0$  are easily proved. The behaviour in the generic case is more complicated.

We summarize now the result of [28], where it is proved that if the *contracting Geometric Lorenz Flow* has an attractor, this attractor is persistent in  $\mathcal{U}$  in a measure theoretical sense (we will make this notion precise below). To do so we start by discussing the properties of the map  $T_0$ . It satisfies the following properties:

- (a)  $T_0$  is onto and piecewise  $C^3$ , with two branches. The order<sup>1</sup> of  $T'_0(x) = O(x^{s-1})$  at  $x = 0$  where  $s = -\frac{\lambda_3}{\lambda_1}$  and  $s - 1 > 0$ ,
- (b)  $T_0$  has a discontinuity at  $x = 0$ ,  $T_0(0^+) = 1/2$ ,  $T_0(0^-) = -1/2$ ,
- (c)  $T'_0(x) < 0$  for every  $x \neq 0$ ,
- (d)  $\max_{x>0} T'_0(x) = T'_0(1/2)$  and  $\max_{x<0} T'_0(x) = T'_0(-1/2)$ ;

there exist  $\rho$  such that also the two following hypothesis are satisfied:

- (e) The points  $1/2$  and  $-1/2$  are preperiodic repelling for  $T_0$ .
- (f) The map  $T_0$  has negative schwarzian derivative.

We choose  $\rho$  such that (e) and (f) are satisfied; under these hypothesis, we have that  $X_0$  has an attractor  $\Lambda_0$ .

Next we precise the meaning of *persistent attractors in a measure theoretical sense*; if not specified, we refer to [28]; the result of Rovella is similar to the results of [5] where it is proved that, for the  $f_a(x) = 1 - ax^2$  there exists a set of parameters of positive Lebesgue measure for which  $f_a$  has an absolutely continuous invariant measure.

Each  $Y$  in a  $C^3$  neighborhood  $\mathcal{U}$  of  $X_0$  preserves a stable foliation and we can construct a transverse section  $\Sigma_Y$  near  $\Sigma$  formed by lines of this stable

<sup>1</sup>we say that  $f(x)$  is  $O(g(x))$  at  $x = x_0$  if there exists  $M, \delta$  such that  $|f(x)| \leq M|g(x)|$  when  $0 < |x - x_0| < \delta$ .

foliation; the first return map  $F_Y$  of  $Y^t$  to  $\Sigma_Y$  has an invariant stable foliation and we can put coordinates  $(x, y)$  in  $\Sigma_Y$  such that  $x = 0$  corresponds to the stable manifold of the singularity,

$$F_Y(x, y) = (T_Y(x), G_Y(x, y))$$

and  $F_Y : [-1/2, 1/2]^2 \rightarrow [-1/2, 1/2]^2$ .

**Definition 2.** Given a subset  $S$  of a finite dimensional Riemannian manifold  $\mathcal{M}$ , we say that  $x$  is a density point of  $S$ , if, denoting by  $m$  the Lebesgue measure on  $\mathcal{M}$ ,  $B_r(x)$  the ball of radius  $r$  and centered at  $x$ , we have:

$$\lim_{r \rightarrow 0} \frac{m(B_r(x) \cap S)}{m(B_r(x))} = 1.$$

**Definition 3.** Given a subset  $S$  of a Banach space  $\mathcal{B}$ , we say that  $x \in S$  is a point of  $k$ -dimensional full density of  $S$  if there exists a  $C^\infty$  submanifold  $\mathcal{N} \subset \mathcal{B}$ , containing  $x$  and having codimension  $k$ , such that for every  $k$ -dimensional manifold  $\mathcal{M}$  intersecting  $\mathcal{N}$  transversally at  $x$ , then  $x$  is a full density point of  $S \cap \mathcal{M}$  in  $\mathcal{M}$ .

**Definition 4.** An attractor  $\Lambda$  of a flow  $X^t \in C^\infty$  is  $k$ -dimensionally almost persistent, if it has a local basin  $U$  such that  $X$  is a  $k$ -dimensional full density point of the set of flows  $Y^t \in C^\infty$  for which  $\Lambda_Y = \cap_{t \geq 0} Y^t(U)$  is an attractor.

Let  $\mathcal{U}$  be a  $C^3$  neighbourhood of  $X_0$  in the space of  $C^\infty$  vector fields, and  $\mathcal{N} = \{Y \in \mathcal{U} \mid \exists k^+, k^- \text{ so that } T_Y^{k^+}(1/2), T_Y^{k^-}(-1/2) \text{ are periodic repelling}\}$ . Note that if  $\mathcal{U}$  is small enough then  $\mathcal{N}$  is a codimension 2 submanifold containing  $X_0$ .

**Theorem 1** ([28]). Let  $\mathcal{M}$  be a 2-dimensional  $C^3$ -submanifold of  $\mathcal{U}$  intersecting  $\mathcal{N}$  transversally, at  $X_0$ . Let  $\{X_a\}$  be a one parameter family of vector fields, contained in  $\mathcal{M}$ , such that the functions  $a \mapsto f_a(\mp 1/2)$  have derivative 1 at  $a = 0$ . Then there is a subset  $E \subset (0, a_0)$  of parameters called the **Rovella parameters**, with  $a_0$  close to 0 and 0 a full density point of  $E$  such that

$$\lim_{a \rightarrow 0} \frac{|E \cap (0, a)|}{a} = 1, \text{ with } \Lambda_{X_a} = \cap_{t \geq 0} X_a^t(U) \text{ an attractor.}$$

This implies that  $\Lambda_0$  is 2-dimensionally almost persistent.

In the remaining of the paper, we restrict ourselves to the setting when  $\{X_a\}$  is one of those one parameter families.

To ease notation we will denote by  $F_a(x, y) = F_{X_a}(x, y)$ ,  $T_a(x) = T_{X_a}(x)$  and  $G_a(x, y) = G_{X_a}(x, y)$  respectively the two dimensional map, the one dimensional map and the map along the stable leaves associated to the Rovella parameter  $a$ . We will denote the eigenvalues of  $X_a$  at the singularity by  $\lambda_{1,a}, \lambda_{2,a}, \lambda_{3,a}$ .

**Lemma 2.2** ([28]). *For all Rovella parameter  $a \in E$  the induced 1-dimensional map  $T_a$  satisfies the following additional properties:*

- (C1)  $T'_a(x) = O(x^{s(a)-1})$  as  $x \rightarrow 0$  where  $s(a) = -\lambda_{3,a}/\lambda_{1,a}$ .  
 (C2) there is  $\lambda_c > 1$  such that for all  $a \in E$ , the points  $1/2$  and  $-1/2$  have Lyapunov exponents greater than  $\lambda_c$ :

$$(T_a^n)'(\pm 1/2) > \lambda_c^n, \quad \text{for all } n \geq 1;$$

- (C3) there is  $\alpha > 0$  such that for all  $a \in E$  the basic assumption holds:

$$|T_a^{n-1}(\pm 1/2)| > e^{-\alpha n}, \quad \text{for all } n \geq 1;$$

- (C4) the forward orbits of the points  $\pm 1/2$  under  $T_a$  are dense in  $[-1/2, 1/2]$  for all  $a \in E$ .

- (C5) for all  $a \in E$ ,  $T_a$  has negative schwarzian derivative, [28, Corollary, pg 243].

**Remark 2.3.** The properties of the Rovella 1-dimensional map are proved in [28]. Item C1 depends on the possibility of linearizing the vector field  $X_a$  near the origin; while this hypothesis is not explicit in [28], this is the reason why we assume  $X_a$  to be a  $C^3$  one parameter family in the space of  $C^\infty$  vector fields, i.e., a  $C^3$  map  $\Xi : \mathbb{R} \supset (-\varepsilon, \varepsilon) \rightarrow C^\infty(\mathbb{R}^3)$ , such that  $\Xi(0) = X_0$ .

We refer to Section 8 for a discussion about the order of the derivatives of  $T_a$  and of  $G_a$ .

The hypothesis of having a  $C^k$  family in the space of  $C^r$  vector fields is widely used in the Rovella-like setting [23, 25].

These properties have strong consequences on the statistical properties of the one dimensional  $T_a$ , as we will relate in Section 3. In our work we prove how the statistical properties of  $T_a$  imply some statistical properties for the flow  $X_a^t$  associated to a Rovella parameter  $a \in E$ .

**Remark 2.4.** The following properties of  $X_a$  for all  $a \in [0, a_0]$ , used in our paper, follow from the properties of the linearization near the origin. We refer to Section 8 for some explicit computations; let  $r(a) = -\lambda_{2,a}/\lambda_{1,a}$  :

- (1)  $\frac{\partial G_a}{\partial y}(x, y) = O(x^{r(a)})$  as  $x$  goes to 0, with  $r(a) > s(a) + 3$  which implies that  $r(a) > 3$ ,
- (2) the order of  $\frac{\partial G_a}{\partial y}(x, y) = O(x^{l(a)})$  and  $l(a) \geq \min(s(a)-1, r(a), r(a)-1)$ , i.e.,  $l(a) \geq s(a) - 1 > 0$
- (3) if  $\ln(x)$  is integrable with respect to the invariant measure  $\mu_a$  of  $T_a$  then the first return time of  $X_a^t$  to  $\Sigma$  is integrable.

### 3. PROPERTIES OF THE ONE DIMENSIONAL ROVELLA MAP

In [21], conditions (C1) and (C3) were used to prove the existence of an ergodic absolutely continuous invariant probability measure for Rovella parameters. In order to obtain uniqueness of that measure, Metzger needed to consider a slightly smaller class of parameters (still with full density at 0), for which conditions (C2) and (C3) imply a strong mixing property. But, in [1], Alves and Soufi deduce the uniqueness of the ergodic absolutely

continuous invariant probability measure for  $a \in E$  not assuming any strong mixing property. Hence, for each  $a \in E$ , the map  $T_a$  has a unique SRB measure  $\mu_a$ .

We now recall some recent results of [1] on the statistical properties of the Rovella one dimensional maps that we will use in our paper.

To state these statistical properties, we start recalling some definitions and facts about  $T_a$ , with  $a$  a Rovella parameter.

**Definition 5.** We say that  $T_a$  is **non-uniformly expanding** if there is a  $c > 0$  such that for Lebesgue almost every  $x \in I$

$$(3.1) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(T'_a(T_a^i(x))) > c.$$

**Definition 6.** We say that  $T_a$  has **slow recurrence to the critical set** if for every  $\epsilon > 0$  there exists  $\delta > 0$  such that for Lebesgue almost every  $x \in I$  it holds

$$(3.2) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} -\log d_\delta(T_a^i(x), 0) \leq \epsilon,$$

where  $d_\delta$  is the  $\delta$ -truncated distance, defined as

$$(3.3) \quad d_\delta(x, y) = \begin{cases} |x - y|, & \text{if } |x - y| \leq \delta, \\ 1, & \text{if } |x - y| > \delta. \end{cases}$$

**Definition 7.** The **expansion time function** is defined as

$$\mathcal{E}_a(x) = \min\{N \geq 1; \frac{1}{n} \sum_{i=0}^{n-1} \log T'_a(T_a^i(x)) > c, \forall n \geq N\},$$

which is well defined and finite almost everywhere in  $I$ , provided (3.1) holds almost everywhere.

Fixing  $\epsilon > 0$  and choosing  $\delta > 0$  conveniently, we define the **recurrence time function**

$$\mathcal{R}_a(x) = \min\{N \geq 1; \frac{1}{n} \sum_{i=0}^{n-1} -\log d_\delta(T_a^i(x), 0) < \epsilon, \forall n \geq N\},$$

which is defined and finite almost everywhere in  $I$ , as long as (3.2) holds almost everywhere.

**Definition 8.** We define the **tail set at time  $n$**  to be the set of points which at time  $n$  have not yet achieved either the uniform exponential growth of the derivative or the uniform slow recurrence:

$$\Gamma_a^n = \{x \in I; \mathcal{E}_a(x) > n \text{ or } \mathcal{R}_a(x) > n\}.$$



**Theorem 2** (Theorem A [1]). Each  $T_a$ , with  $a \in E$ , is non-uniformly expanding and has slow recurrence to the critical set. Moreover, there are  $C > 0$  and  $\tau > 0$  such that for all  $a \in E$  and  $n \in \mathbb{N}$ , it holds

$$(3.4) \quad |\Gamma_a^n| \leq Ce^{-\tau n}.$$

In [1] the authors deduced several interesting consequences from (3.4), which follow from Theorem 2 and Theorem 3 in the seminal paper [32]. Their results involve the class of Hölder continuous function<sup>2</sup> with a given exponent  $\alpha$ , denoted by  $H(\alpha)$ . This exponent  $\alpha$  depends on the distortion properties of the induced map and, as in [1], we will only state that it exists and not worry about any more precise estimate.

The main result in [1] is that for all Rovella parameter  $a \in E$ :

- (1)  $T_a$  has a unique ergodic absolutely continuous invariant probability measure  $\mu_a$ ;
- (2) the measure  $\mu_a$  has exponential decay of correlations for  $H(\alpha)$ -observables against  $L^\infty(\mu_a)$  observables.

From (3.4) and [32] follows directly the next Corollary, used in our treatment.

**Corollary 3.1.** *For all  $a \in E$ , let  $\mu_a$  be absolutely continuous invariant measure of  $T_a$ . Let  $f \in H(\alpha)$  and let  $g \in L^\infty(\mu_a)$  then  $T_a$  satisfies **exponential convergence to equilibrium**, i.e., there exist a constant  $C_{f,g}$  (which may depend on  $f$  and  $g$ ), and a constant  $\Lambda < 1$  such that*

$$\left| \int f \cdot (g \circ T^n) dm - \int g d\mu_a \int f dm \right| \leq C_{f,g} \Lambda^n.$$

To apply our strategy we need a different definition of convergence to equilibrium, expressing the constant in function of the norms of the observables  $f$  and  $g$ .

**Definition 3.2.** We say that  $(F, \mu)$  has exponential convergence to equilibrium with respect to norms  $\|\cdot\|_a$  and  $\|\cdot\|_b$ , if there are  $C, \Lambda \in \mathbb{R}^+$ ,  $\Lambda < 1$  such that for  $f \in L^1(m), g \in L^1(m)$

$$\left| \int f \cdot (g \circ T^n) dm - \int g d\mu \int f dm \right| \leq C \Lambda^n \cdot \|g\|_a \cdot \|f\|_b, \quad n \geq 1.$$

We can obtain this uniform estimation on the convergence to equilibrium using the uniform boundedness principle and an observation which can be found in [7, Appendix B]:

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<sup>2</sup>we will denote by

$$\text{Höl}_\alpha(f) = \sup_{x, y \in \mathbb{I}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

and by  $\|f\|_{H(\alpha)} := \|f\|_\infty + \text{Höl}_\alpha(f)$  the  $\alpha$ -Hölder norm.

**Proposition 3.3.**  $(T_a, \mu)$  has exponential convergence to equilibrium in the following sense. There are  $C, \Lambda \in \mathbb{R}^+$ ,  $\Lambda < 1$  such that

$$\left| \int f \cdot (g \circ F^n) dm - \int g d\mu_a \int f dm \right| \leq C \Lambda^n \cdot \|g\|_\infty \cdot \|f\|_{H(\alpha)}$$

*Proof.* Let  $B_{L^\infty}$  be the unit ball in  $L^\infty$ . For any  $n \in \mathbb{N}$  and  $f$  such that  $\|f\|_{H(\alpha)} \leq 1$ , let us consider the family of non-negative functions defined by

$$q_{(n,f)}(g) = \Lambda^n \left| \int f g \circ T^n dm - \int g d\mu_a \int f dm \right|.$$

For any  $g_1, g_2, g_3 \in L^\infty$  it holds

$$q_{(n,f)}(g_1 + g_2) \leq q_{(n,f)}(g_1) + q_{(n,f)}(g_2)$$

and

$$q_{(n,f)}(-g_3) = q_{(n,f)}(g_3).$$

Moreover by Corollary 3.1, for each  $g \in L^\infty$

$$\sup_{n \in \mathbb{N}} q_{(n,f)}(g) < \infty.$$

By the uniform boundedness principle then (see [18], Theorem 1.29, page 139) it follows

$$\sup_{n \in \mathbb{N}, \|g\|_\infty \leq 1} q_{(n,f)}(g) := C_f < \infty.$$

And we can write

$$\left| \int f \cdot (g \circ F^n) dm - \int g d\mu_a \int f dm \right| \leq C_f \Lambda^n \cdot \|g\|_\infty.$$

Now applying the same idea one more time we can uniformize on  $f$  and obtain the statement (see [7] for the details).  $\square$

In the next section we will see how a result on the convergence to equilibrium for a certain map can be extended to a skew product with contracting fibers, based on that map.

#### 4. GENERAL RESULT ON CONVERGENCE AND CORRELATION DECAY FOR SKEW-PRODUCTS WITH CONTRACTING FIBERS

We recall some general results from [2]. For this, let  $\mathbb{I} = [-1/2, 1/2]$  the unit interval and denote by  $\mathbb{Q} = \mathbb{I} \times \mathbb{I}$ . We consider maps  $F : \mathbb{Q} \rightarrow \mathbb{Q}$  preserving a regular foliation, which contracts the leaves and whose quotient map (the induced map on the space of leaves) has exponential convergence to equilibrium. We will give an estimation of the speed of convergence to equilibrium for this kind of maps, establishing that it is also exponential.

**Definition 4.1.** If  $f : \mathbb{Q} \rightarrow \mathbb{Q}$  is integrable, we denote by  $\pi(f) : \mathbb{I} \rightarrow \mathbb{I}$  the function  $\pi(f) : x \mapsto \int_{\mathbb{I}} f(x, t) dt$ .

Let us consider the following anisotropic norm, considering Lipschitz regularity only on the vertical direction. Let  $\|\cdot\|_{\uparrow\text{Lip}}$  be defined by

$$(4.1) \quad \|g\|_{\uparrow\text{Lip}} = \|g\|_{\infty} + \text{Lip}_y(g),$$

where for simplicity we consider

$$(4.2) \quad \|g\|_{\infty} := \sup_{x,y \in \mathbb{I}} |g(x,y)| \quad \text{and} \quad \text{Lip}_y(g) := \sup_{\substack{x,y_1,y_2 \in \mathbb{I} \\ y_1 \neq y_2}} \frac{|g(x,y_2) - g(x,y_1)|}{|y_2 - y_1|}.$$

The following is proved in [2, Theorem 1]

**Theorem 4.2.** *Let  $F : \mathbb{Q} \circlearrowleft$  be a Borel function such that  $F(x,y) = (T(x), G(x,y))$ . Let  $\mu$  be an  $F$ -invariant measure with absolutely continuous marginal  $\mu_T$  on the  $x$ -axis which, moreover, is  $T$ -invariant. Let us suppose that*

- (1)  $(T, \mu_T)$  has exponential convergence to equilibrium with respect to the norm  $\|\cdot\|_{\infty}$  (the  $L^{\infty}$  norm) and to a norm which we denote by  $\|\cdot\|_{-} \geq \|\cdot\|_{\infty}$ .
- (2)  $T$  is nonsingular with respect to the Lebesgue measure, piecewise continuous and monotonic: there is a collection of intervals  $\{I_i\}_{i=1,\dots,m}$ ,  $\cup I_i = I$  such that on each  $I_i$ ,  $T$  is an homeomorphism onto its image.
- (3)  $F$  is a contraction on each vertical leaf:  $G$  is  $\lambda$ -Lipschitz in  $y$  with  $\lambda < 1$ .

Then  $(F, \mu)$  has exponential convergence to equilibrium in the following sense. There are  $C, \Lambda \in \mathbb{R}^+$ ,  $\Lambda < 1$  such that

$$\left| \int f \cdot (g \circ F^n) d\mu - \int g d\mu \int f d\mu \right| \leq C \Lambda^n \cdot \|g\|_{\uparrow\text{Lip}} \cdot (\|\pi(f)\|_{-} + \|f\|_1)$$

for each  $f \geq 0$ .

Now let us relate convergence to equilibrium to decay of correlations. This will be done by the following statement (see [2, Theorem 2 and 3])

**Theorem 4.3.** *Let  $F : \mathbb{Q} \circlearrowleft$  be a Borel function such that  $F(x,y) = (T(x), G(x,y))$ ,  $\mu$  an  $F$ -invariant probability measure with absolutely continuous  $T$ -invariant marginal  $\mu_T$  on the  $x$ -axis and satisfying*

- (1)  $(T, \mu_T)$  has exponential convergence to equilibrium with respect to the norm  $\|\cdot\|_{\infty}$  and to a norm  $\|\cdot\|_{-}$ ;
- (2)  $T$  is nonsingular with respect to the Lebesgue measure, piecewise continuous and monotonic: there is a collection of intervals  $\{I_i\}_{i=1,\dots,m}$ ,  $\cup I_i = \mathbb{I}$  such that on each  $I_i$ ,  $T$  is an homeomorphism onto its image.
- (3)  $F$  is a uniform contraction on each vertical leaf.
- (4) Moreover, let us assume that there are  $C, K \in \mathbb{R}$  and a seminorm  $\|\cdot\|_{\square}$  such that

$$\|\pi(f \circ F^n)\|_{-} + \|f \circ F^n\|_{\square} \leq C_1 K^n (\|\pi(f)\|_{-} + \|f\|_{\uparrow\text{Lip}} + \|f\|_{\square}), \quad \forall n \geq 1.$$

Then  $F$  has exponential decay of correlations: there are  $C, \Lambda \in \mathbb{R}^+$ ,  $\Lambda < 1$

$$\left| \int f \cdot (g \circ F^n) d\mu - \int g d\mu \int f d\mu \right| \leq C_2 \Lambda^n \|g\|_{\uparrow Lip} (\|f\|_{\uparrow Lip} + \|\pi(f)\|_- + \|f\|_{\square})$$

for all  $f, g : \mathbb{Q} \rightarrow \mathbb{R}$  and  $n \geq 0$ .

## 5. DECAY OF CORRELATIONS FOR ADAPTED FUNCTION SPACES

Many results about decay of correlations and convergence to equilibrium are obtained for Hölder observables. Yet, in systems with discontinuities, this class of functions is not the most natural, since it is not preserved by the transfer operator. We show how to extend those results to generalized bounded variation observables. This extension is necessary to apply Theorem 4.3.

**5.1. Functions of bounded  $p$ -variation.** We recall the main definitions and basic results about bounded  $p$ -variation (see [20]).

Given a function we define its universal  $p$ -Variation as the following adaptation of the usual notion of bounded variation.

**Definition 5.1.** Let  $g : [0, 1] \rightarrow \mathbb{R}$  and let:

$$\text{Var}_p(g, x_1, \dots, x_n) = \left( \sum_{i \leq n} |g(x_i) - g(x_{i+1})|^p \right)^{\frac{1}{p}}.$$

The **universal  $p$ -Variation** is:

$$\text{Var}_p(g) = \sup_{\mathcal{P}} \text{Var}_p(g, x_1, \dots, x_n),$$

where  $\mathcal{P}$  is the collection of all the finite subdivisions of  $[0, 1]$ . Let

$$\text{UBV}_p = \{g : \text{Var}_p(g) < \infty\}$$

be the space of functions of bounded universal  $p$ -Variation.

In the following  $m$  be the Lebesgue measure on the unit interval,  $\epsilon > 0$  and  $h : [0, 1] \rightarrow \mathbb{C}$ . We define

$$\text{osc}(h, \epsilon, x) = \text{ess sup}\{|h(y_1) - h(y_2)| : y_1, y_2 \in B_\epsilon(x)\},$$

where  $B_\epsilon(x)$  is the ball centered in  $x$  with radius  $\epsilon$ , and the essential supremum is taken with respect to the Lebesgue measure. Now let us define

$$\text{osc}_p(h, \epsilon) = \|\text{osc}(h, \epsilon, x)\|_p, \quad 1 \leq p \leq \infty,$$

where the  $p$ -norm is taken with respect to  $m$ .

**Remark 5.2.**  $\text{osc}_p(h, *) : (0, A] \rightarrow [0, \infty]$  is a non decreasing function and  $\text{osc}_p(h, \epsilon) \geq \text{osc}_1(h, \epsilon)$ .

Fixed  $0 \leq r \leq 1$ , set  $R_{p,r,n} = \{h | \forall \epsilon \in (0, A], \text{osc}_p(h, \epsilon) \leq n\epsilon^r\}$  and  $S_{p,r} = \cup_{n \in \mathbb{N}} R_{p,r,n}$ .

We can now define:

- (1)  $BV_{p,r}$  as the space of  $m$ -equivalence classes of functions in  $S_{p,r}$
- (2)  $\text{Var}_{p,r}(h) = \sup_{0 < \epsilon \leq A} (\epsilon^{-r} \text{osc}_p(h, \epsilon))$  (we remark that this definition depends on a fixed constant  $A$  and that  $\text{Var}_{p,r}(h) \geq \text{Var}_{1,r}(h)$ ).
- (3) for  $h \in BV_{p,r}$  we define  $\|h\|_{p,r} := \text{Var}_{p,r}(h) + \|h\|_p$ .

It turns out that  $BV_{p,r}$  with the norm  $\|h\|_{p,r}$  is a Banach space; see [20, Thm. 1.13]. In the following we will fix  $A = 1$ .

**Proposition 5.3.**  *$UBV_p \subseteq BV_{p,\frac{1}{p}} \subseteq BV_{1,\frac{1}{p}}$  for all  $1 \leq p < \infty$ . Moreover*

$$(5.1) \quad \text{Var}_{1,\frac{1}{p}}(h) \leq \text{Var}_{p,\frac{1}{p}}(h) \leq 2^{\frac{1}{p}} \text{Var}_p(h).$$

In what follows we need to compare the  $\|\cdot\|_{p,r}$  norm with the  $L^\infty(m)$  norm. The following Lemma will be useful (see [2, Lemma 2]).

**Lemma 5.4.** *If  $f \in BV_{1,r}$  ( $r \leq 1$ ), then  $f \in L^\infty(m)$  and*

$$\|f\|_\infty \leq A^{r-1} \cdot \|f\|_{1,r},$$

where  $A$  is the constant in the definition of  $\|\cdot\|_{1,r}$  (see item 2 above).

**5.2. Hölder convergence to equilibrium implies convergence to equilibrium for bonded  $p$ -Variation.** Suppose we have a system having exponential convergence to equilibrium with  $H(\alpha)$  and  $L^\infty$  observables, let us estimate the convergence for  $f \in BV_{1,p}$  and  $g \in L^\infty$ .

**Proposition 5.5.** *If for each  $f \in H(\alpha)$  and  $g \in L^\infty$  we have convergence to equilibrium with speed  $\Phi$  :*

$$\left| \int g \circ T^n f \, dm - \int g \, d\mu \int f \, dm \right| \leq \|g\|_\infty \|f\|_{H(\alpha)} \Phi(n),$$

then for each  $f, g$  respectively in  $BV_{1,\alpha}$  and  $L^\infty$  it holds

$$\left| \int g \circ T^n f \, dm - \int g \, d\mu \int f \, dm \right| \leq \|g\|_\infty \|f\|_{1,\alpha} 6\sqrt{\Phi(n)}.$$

*Proof.* Let us consider  $\rho_\epsilon = \frac{1}{2\epsilon} 1_{B(0,\epsilon)}$  a multiple of the characteristic function of an interval of radius  $\epsilon$ , small.

Let us consider  $f \in BV_{1,\alpha}$ , approximate  $f$  with  $f_\epsilon = f * \rho_\epsilon$  (the convolution with  $\rho_\epsilon$ ) and estimate the integral:

$$\begin{aligned} \left| \int g \circ T^n f \, dm \right| &\leq \left| \int g \circ T^n (f + f_\epsilon - f_\epsilon) \, dm - \int g \, d\mu \int f + f_\epsilon - f_\epsilon \, dm \right| \\ &\leq \int |g \circ T^n (f - f_\epsilon)| \, dm + \int g \, d\mu \int |f - f_\epsilon| \, dm \\ &\quad + \left| \int g \circ T^n f_\epsilon \, dm - \int g \, d\mu \int f_\epsilon \, dm \right| \\ &\leq 2\|g\|_\infty \|f - f_\epsilon\|_1 + \|g\|_\infty \|f_\epsilon\|_{H(\alpha)} \Phi(n). \end{aligned}$$

Now let us estimate  $\|f - f_\epsilon\|_1$

$$\begin{aligned} \|f - f * \rho_\epsilon\|_1 &\leq \int_I \left| \int_{B(0,\epsilon)} [f(x-y) - f(x)] \rho_\epsilon(y) dy \right| dx \\ &\leq \int_I \sup_{y \in B(x,\epsilon)} |f(x-y) - f(x)| dx \\ &\leq \text{osc}_1(f, \epsilon). \end{aligned}$$

We bound the Hölder seminorm of  $f_\epsilon$

$$\text{Höl}_\alpha(f * \rho_\epsilon(x)) = \sup_{x_1, x_2 \in I} |x_1 - x_2|^{-\alpha} \left| \int_{B(0,\epsilon)} [f(x_1 - y) - f(x_2 - y)] \rho_\epsilon(y) dy \right|,$$

by the definition of  $\rho_\epsilon$

$$\left| \int_{B(0,\epsilon)} [f(x_1 - y) - f(x_2 - y)] \rho_\epsilon(y) dy \right| \leq \begin{cases} 2\epsilon^{-1} |x_1 - x_2| \|f\|_\infty & \text{if } |x_1 - x_2| \leq \epsilon \\ 2\|f\|_\infty & \text{if } |x_1 - x_2| > \epsilon \end{cases}.$$

Hence

$$\text{Höl}_\alpha(f * \rho_\epsilon(x)) \leq \sup_{x_1, x_2 \in I} \left( \begin{cases} |x_1 - x_2|^{-\alpha+1} 2\epsilon^{-1} \|f\|_\infty & \text{if } |x_1 - x_2| \leq \epsilon \\ |x_1 - x_2|^{-\alpha} 2\|f\|_\infty & \text{if } |x_1 - x_2| \geq \epsilon \end{cases} \right).$$

It follows

$$\text{Höl}_\alpha(f * \rho_\epsilon(x)) \leq 2\epsilon^{-\alpha} \|f\|_\infty \leq 2\epsilon^{-\alpha} \|f\|_{1,\alpha}.$$

Summarizing

$$\begin{aligned} \left| \int g \circ T^n f dm - \int g d\mu \int f dm \right| &\leq \\ &\leq 2\|g\|_\infty \|f - f_\epsilon\|_1 + \|g\|_\infty \|f_\epsilon\|_{H(\alpha)} \Phi(n) \\ &\leq \|g\|_\infty (2\text{osc}_1(f, \epsilon) + 4\epsilon^{-\alpha} \|f\|_{1,\alpha} \Phi(n)) \\ &\leq \|g\|_\infty (2\epsilon^\alpha \|f\|_{1,\alpha} + 4\epsilon^{-\alpha} \|f\|_{1,\alpha} \Phi(n)) \\ &= \|g\|_\infty \|f\|_{1,\alpha} (2\epsilon^\alpha + 4\epsilon^{-\alpha} \Phi(n)). \end{aligned}$$

For each  $n$  we can take  $\epsilon$  such that  $\epsilon^{2\alpha} = \Phi(n)$  we have  $\epsilon^\alpha = \sqrt{\Phi(n)}$  and

$$\left| \int g \circ T^n f dm - \int g d\mu \int f dm \right| \leq 6\|g\|_\infty \|f\|_{1,\alpha} \sqrt{\Phi(n)}.$$

□

By this proposition it follows that if a system has exponential decay of correlation with respect to Hölder observables, it has also exponential decay with respect to generalized bounded variation ones.

## 6. DECAY OF CORRELATIONS FOR THE TWO DIMENSIONAL ROVELLA MAP: PROOF OF THEOREM A

In the following section we explain which are the norms involved, in our case, in the statements of Section 4 and we will check that the 2-dimensional Rovella maps satisfies the hypothesis of our theorem, implying the proof of Theorem A.

As showed in Proposition 3.3 the one-dimensional Rovella map for a Rovella parameter  $a \in E$  satisfies exponential convergence to equilibrium with respect to Hölder and  $L^\infty$  observables. By using the results in Section 5 and in particular by Proposition 5.5 we have the following.

**Corollary 6.1.** *If  $a$  is a Rovella parameter, the one dimensional Rovella map satisfies exponential convergence to equilibrium with respect to the norms  $\|\cdot\|_{1,\alpha}$  and  $\|\cdot\|_\infty$ .*

We will need another definition of variation for maps with two variables. Similarly to the one dimensional case, if  $f : \mathbb{Q} \rightarrow \mathbb{R}$  and  $x_i \leq x_2 \leq \dots \leq x_n$ , let us define

$$\text{Var}^\square(f, x_1, \dots, x_n, y_1, \dots, y_n) = \sum_{1 \leq i \leq n} |f(x_i, y_i) - f(x_{i+1}, y_i)|.$$

We then consider the supremum  $\text{Var}^\square(f, x_1, \dots, x_n, y_1, \dots, y_n)$  over all subdivisions  $x_i$  and all choices of the  $y_i$

$$\text{Var}^\square(f) = \sup_n \left( \sup_{(x_i \leq x_2 \leq \dots \leq x_n) \in \mathcal{I}, (y_i) \in \mathcal{I}} \text{Var}^\square(f, x_1, \dots, x_n, y_1, \dots, y_n) \right).$$

We want to apply Theorem 4.3 using  $\|\cdot\|_{1,\alpha}$  as the norm  $\|\cdot\|_-$  and  $\text{Var}^\square(\cdot)$  as the seminorm  $\|\cdot\|_\square$ . Thus we need to prove that item 4 is satisfied for this seminorm; to do so, thanks to [2, Lemma 16] the only thing we need to prove now is the following.

**Lemma 6.2.** *For each  $a \in [0, a_0]$  we have that  $\text{Var}^\square(G_a) < \infty$*

*Proof.* Remembering the definition:

$$\text{Var}^\square(G_a, x_1, \dots, x_n, y_1, \dots, y_n) = \sum_n |G_a(x_i, y_i) - G_a(x_{i+1}, y_i)|.$$

By Remark 2.4 we have that  $\frac{\partial G_a}{\partial x}$  is bounded for all  $x \in I$  and there exists  $M$  such that

$$\frac{\partial G_a}{\partial x} \leq M|x|^{s(a)-1},$$

with  $s(a) - 1 > 0$ . Moreover, we can observe that, if  $x \leq 0$  and  $\tilde{x} > 0$  we have that

$$|G(x, y) - G(\tilde{x}, y)| < 1.$$

Therefore, if  $-1/2 = x_1 < \dots < x_k < 0 < x_{k+1} < \dots < x_n = 1/2$  then

$$\begin{aligned} & \sum_0^{k-1} |G(x_i, y_i) - G(x_{i+1}, y_i)| + |G(x_k, y_i) - G(x_{k+1}, y_i)| \\ & \quad + \sum_{k+1}^{n-1} |G(x_i, y_i) - G(x_{i+1}, y_i)| \leq \\ & \sum_0^{k-1} \left| \frac{\partial G}{\partial x}(\xi_i) \right| |x_{i+1} - x_i| + 1 + \sum_{k+1}^{n-1} \left| \frac{\partial G}{\partial x}(\xi_i) \right| |x_{i+1} - x_i| \leq 1 + M, \end{aligned}$$

where  $\xi_i \in [x_i, x_{i+1}]$  for  $i \neq k$ .  $\square$

By Theorems 4.2 and 4.3, this gives the following consequence:

**Proposition 6.3.** *If  $a$  is a Rovella parameter, the two dimensional Rovella map  $F_a$  has exponential convergence to equilibrium and exponential decay of correlations: There are  $C, \Lambda \in \mathbb{R}^+$ ,  $\Lambda < 1$  such that for  $n \geq 1$ :*

$$\begin{aligned} & \left| \int f \cdot (g \circ F^n) d\mu - \int g d\mu \int f d\mu \right| \leq C\Lambda^n \cdot \|g\|_{\uparrow Lip} \cdot (\|\pi(f)\|_{1,\alpha} + \|f\|_1) \\ & \left| \int f \cdot (g \circ F^n) d\mu - \int g d\mu \int f d\mu \right| \leq C\Lambda^n \cdot \|g\|_{\uparrow Lip} (\|f\|_{\uparrow Lip} + \|\pi(f)\|_{1,\alpha} + Var^\square f). \end{aligned}$$

From this, as in [2] follows the decay of correlation with respect to Lipschitz observables, proving Theorem A.

## 7. QUANTITATIVE RECURRENCE AND LOGARITHM LAW, PROOF OF THEOREM B AND C

In this section we will show some consequences of the decay of correlations proved above. We show how these results imply hitting time and quantitative recurrence estimations.

Let us consider a discrete time dynamical system  $(X, T, \mu)$ , where  $(X, d)$  is a metric space and  $T : X \rightarrow X$  is a measurable map preserving a finite measure  $\mu$ . Let us consider two points  $x, y$  in  $X$  and the time which is necessary for the orbit of  $x$  to approach  $y$  at a distance less than  $r$

$$\tau_r(x, y) = \min\{n \in \mathbb{N}^+ : d(T^n(x), y) < r\};$$

if  $x = y$  we will use the notation  $\tau_r(x) := \tau_r(x, x)$ .

We consider the behavior of  $\tau_r(x, y)$  as  $r \rightarrow 0$ . In many interesting cases this is a power law  $\tau_r(x, y) \sim r^R$ . When  $x \neq y$  the exponent is a quantitative measure of how fast the orbit starting from  $x$  approaches a point  $y$ . When  $x = y$  the exponent  $R$  gives a quantitative measure of the speed of recurrence of an orbit near to its starting point, and this will be a quantitative recurrence indicator.



**Definition 9.** Let  $\nu$  be a measure on  $X$ , metric space. The upper and lower local dimension are defined as:

$$\bar{d}_\nu(x) = \limsup_{r \rightarrow 0} \frac{\log(\mu(B_r(x)))}{\log(r)} \quad \underline{d}_\nu(x) = \liminf_{r \rightarrow 0} \frac{\log(\mu(B_r(x)))}{\log(r)}.$$

If  $\bar{d}_\nu(x) = \underline{d}_\nu(x)$  we say the local dimension exists at  $x$  and denote it by  $d_\nu(x) = \underline{d}_\nu(x)$ ; if the local dimension exists and  $d_\nu(x)$  is constant for  $\nu$  a.e.  $x$ , we say the system is exact dimensional.

The results of [29] and [9], give a quantitative estimation for these indicators for rapidly mixing systems, and can be summarized in the following theorem (see also [11]) that directly implies Theorem B.

**Theorem 7.1.** *Let  $(X, T, \nu)$  be a measure preserving system with a decay of correlations with respect to Lipschitz observables faster than any polynomial rate. Let  $x, y \in X$ ,*

- (1) *if the local dimension  $d_\nu(y)$  exists then*

$$\limsup_{r \rightarrow 0} \frac{\log(\tau_r(x, y))}{-\log(r)} = \liminf_{r \rightarrow 0} \frac{\log(\tau_r(x, y))}{-\log(r)} = d_\nu(y)$$

*for  $\nu$ -almost every  $x$ .*

- (2) *If  $X \subseteq \mathbb{R}^d$  for some  $d$ , then*

$$\limsup_{r \rightarrow 0} \frac{\log(\tau_r(x))}{-\log(r)} = \bar{d}_\nu(x) \quad \text{and} \quad \liminf_{r \rightarrow 0} \frac{\log(\tau_r(x))}{-\log(r)} = \underline{d}_\nu(x)$$

*for  $\nu$ -almost every  $x$  such that  $\underline{d}_\nu(x) > 0$ .*

*Proof of Theorem B.* Observe that for a Rovella two dimensional map  $F_a$ ,  $a \in E$  a Rovella parameter, the SRB measure  $\mu_{F_a}$  has exponential decay of correlations with respect to Lipschitz observables and that  $\underline{d}_{\mu_{F_a}}(x) > 0$  almost everywhere (the projection of the measure on the basis is absolutely continuous). Applying Theorem 7.1 we have Theorem B.  $\square$

Now we extend the result to the Rovella flow, following [12, 8]. We need to check that the return time to the section is integrable.

**Proposition 7.2.** *Let  $a \in E$  be a Rovella parameter, and let  $\mu_a$  be the invariant measure for the one dimensional map  $T_a$  then:*

$$\int_{-1/2}^{1/2} -\log|x| d\mu_a \leq \infty.$$

*Proof.* In [1] it was proved that for each Rovella parameter the map  $T_a$  has slow recurrence to the critical set (cit. Def. 6) we will use this to give a bound on the integral of  $\log|x|$ .

First of all, let  $I_\delta := (-\delta, \delta)$  and let  $J_\delta = I \setminus I_\delta$ :

$$\int_I -\log|x| d\mu_a = \int_{J_\delta} -\log|x| d\mu_a + \int_{I_\delta} -\log|x| d\mu_a;$$

first we will bound

$$\int_{J_\delta} -\log|x|d\mu_a \leq \int_{J_\delta} -\log(\delta)d\mu_a \leq \int_I -\log(\delta)d\mu_a = -\log(\delta).$$

Now we observe that, if we denote by  $\chi_{I_\delta}$  the characteristic function of  $I_\delta$  and by  $d_\delta$  the truncated distance as in Def. 6:

$$-\log|x| \cdot \chi_{I_\delta} = -\log(d_\delta(x, 0)),$$

thus

$$\int_{I_\delta} -\log|x|d\mu_a = \int_I -\log|x| \cdot \chi_{I_\delta}d\mu_a = \int_I -\log(d_\delta(x, 0))d\mu_a.$$

Let now  $\phi_k(x) = \min\{-\log(d_\delta(x, 0)), k\}$ ; trivially  $\|\phi_k\|_\infty \leq k$ , therefore  $\phi_k \in L^1(\mu_a)$  and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi_k(T_a^i(x)) = \int_I \phi_k d\mu_a,$$

for  $\mu_a$  a.e.  $x$ . Moreover, for all  $x$ ,  $k$  and  $n$ :

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi_k(T_a^i(x)) \leq \frac{1}{n} \sum_{i=0}^{n-1} -\log(d_\delta(T_a^i(x), 0)).$$

We argue by contradiction; suppose  $\int_I -\log(d_\delta(x, 0))d\mu_a = +\infty$ ; therefore

$$\begin{aligned} +\infty &= \lim_{k \rightarrow +\infty} \int_I \phi_k(x)d\mu_a = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi_k(T_a^i(x)) \quad \mu_a - a.e. \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} -\log(d_\delta(T_a^i(x), 0)) \leq \epsilon < +\infty \end{aligned}$$

since the slow recurrence to the critical set holds for a set of full Lebesgue measure.  $\square$

**Remark 7.3.** In particular, since the return time on the section for the Geometric Rovella Flow is controlled by  $-\log|x|/\lambda_1$  this implies that the return time on the section for the Geometric Rovella Flow is integrable.

**Remark 7.4.** Proposition 7.2 is needed in the Rovella case because the density of  $\mu_a$  may be unbounded near  $x = 0$ ; this does not happen in the Lorenz case since we have stronger results concerning the regularity of the density of the invariant measure of the one dimensional map that cannot be applied in the non uniformly hiperbolic case.

We show now that the return time on the section is integrable for all the Rovella parameters.

**Proposition 7.5.** *If the neighborhood  $\mathcal{U}$  is small enough, then the return time to the section is integrable for each Rovella parameter  $a \in E$ .*

*Proof.* Due to the linearization argument in Section 8, we know that there exist two constants  $C_1$  and  $C_2$  such that the return time to the cross section  $\tau_X$  satisfies:

$$C_1 - \frac{\log |x|}{\tilde{\lambda}_1} \leq \tau_X(x, y) \leq C_2 - \frac{\log |x|}{\tilde{\lambda}_1}.$$

Since  $-\log |x|$  is integrable for each Rovella parameter, we have the thesis.  $\square$

Consider a flow  $X^t$  in  $\mathbb{R}^3$  having a transversal section  $\Sigma$  whose first return time is integrable. As before let  $F : \Sigma \setminus \Gamma \rightarrow \Sigma$  be the first return map associated. Let  $\mu_F$  be an ergodic invariant measure for  $F$ . Now, if we consider a flow having such a map as its Poincaré section and integrable return time, we can construct an SRB invariant measure  $\mu_X$  for the flow. Let  $x, x_0 \in \mathbb{R}^3$  and

$$\tau_r^{X^t}(x, x_0) = \inf\{t \geq 0 | X^t(x) \in B_r(x_0)\}$$

be the time needed for the  $X$ -orbit of a point  $x$  to enter for the *first time* in a ball  $B_r(x_0)$ . The number  $\tau_r^{X^t}(x, x_0)$  is the *hitting time associated* to the flow  $X^t$  and  $B_r(x_0)$ .

If the orbit  $X^t$  starts at  $x_0$  itself let us consider the return time in the ball and denote

$$(7.1) \quad \tau_r^X(x_0) = \inf\{t \in \mathbb{R}^+ : X^t(x_0) \in B_r(x_0), \exists i < t, s.t. X^i(x_0) \notin B_r(x_0)\}.$$

If  $x, x_0 \in \Sigma$  and  $B_r^\Sigma(x_0) = B_r(x_0) \cap \Sigma$  we denote

$$\tau_r^\Sigma(x, x_0) = \min\{n \in \mathbb{N}^+ ; F^n(x) \in B_r^\Sigma(x_0)\}$$

the *hitting time associated* to the discrete system  $F$ .

Given any  $x$  we recall that we denote with  $t(x)$  the first strictly positive time, such that  $X^{t(x)}(x) \in \Sigma$  (the *return time* of  $x$  to  $\Sigma$ ). A relation between  $\tau_r^X(x, x_0)$  and  $\tau_r^\Sigma(x, x_0)$  is proved in [8] (Proposition 5.2). Let  $x \in \mathbb{R}^3$  and  $\pi(x)$  be the projection on  $\Sigma$  given by  $\pi(x) = y$  if  $x$  is on the orbit of  $y \in \Sigma$  and the orbit from  $y$  to  $x$  does not cross  $\Sigma$  (if  $x \in \Sigma$  then  $\pi(x) = x$ ).

**Proposition 7.6.** *Under the assumptions listed above, there is a full  $\mu_X$  measure set  $B \subset \mathbb{R}^3$  such that if  $x_0 \in \mathbb{R}^3$  is regular and  $x \in B$  it holds (provided the second limits exist)*

$$(7.2) \quad \lim_{r \rightarrow 0} \frac{\log \tau_r^{X^t}(x, x_0)}{-\log r} = \lim_{r \rightarrow 0} \frac{\log \tau_r^\Sigma(\pi(x), \pi(x_0))}{-\log r}.$$

By Theorem B, applying this last proposition to the 2-dimensional system  $(\Sigma, F_a, \mu_a)$  and the contracting geometric Lorenz flow and its invariant SRB measure, we get the logarithm law and the quantitative recurrence statement for the flow, i.e., Theorem C.

## 8. APPENDIX: LINEARIZATION AND PROPERTIES OF THE POINCARÉ MAP

Please note that to prove some our result we use a linearization near the singularity of the flow; to do so, we need a  $C^1$  linearization result; we will use Theorem 7.1 page 257 of [15] in its version for flows.

**Theorem 8.1.** *Let  $n \in \mathbb{Z}^+$  be given. Then there exists an integer  $N = N(n) \geq 2$  such that: if  $\Gamma$  is a real non-singular  $d \times d$  matrix with eigenvalues  $\gamma_1, \dots, \gamma_d$  satisfying*

$$\sum_{i=1}^d m_i \gamma_i \neq \gamma_k \quad \text{for all } k = 1, \dots, d \text{ and } 2 \leq \sum_{j=1}^d m_j \leq N(n)$$

*and if  $\dot{\xi} = \Gamma(\xi) + \Xi(\xi)$  and  $\dot{\zeta} = \Gamma\zeta$ , where  $\xi, \zeta \in \mathbb{R}^d$  and  $\Xi$  is of class  $C^N$  for small  $\|\xi\|$  with  $\Xi(0) = 0$ ,  $\partial_\xi \Xi(0) = 0$ ; then there exists a  $C^n$  diffeomorphism  $R$  from a neighborhood of  $\xi = 0$  to a neighborhood of  $\zeta = 0$  such that  $R\xi(t)R^{-1} = \zeta(t)$  for all  $t \in \mathbb{R}$  and initial conditions for which the flows  $\zeta(t)$  and  $\xi(t)$  are defined in the corresponding neighborhood of the origin.*

Since the resonance conditions are open, there exists an  $N = N(1)$  such that, if we choose the eigenvalues of the Geometric Rovella Flow respecting the resonance conditions, there exists a  $C^N$  neighborhood of  $X_0$  such that all the vector fields in the neighborhood respect the resonance conditions and can be  $C^1$ -linearized. Generically, the linear part of a vector field  $\tilde{X}$  in such a neighborhood is different from the linear part of  $X_0$ ; our aim is not to find a common linearization for all the fields in the neighborhood but to ensure the fact that  $\tilde{X}$  is  $C^1$ -linearizable.

**8.1. Behaviour near the fixed point.** Let  $\tilde{X}$  be in a  $C^N$  neighborhood of  $X_0$  such that the resonance condition are still satisfied; then  $\tilde{X}$  can be  $C^1$ -linearized.

Denote by  $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3$  the eigenvalues of  $\tilde{X}$  at the fixed point, and denote by  $\tilde{r} = -\tilde{\lambda}_2/\tilde{\lambda}_1$  and by  $\tilde{s} = -\tilde{\lambda}_3/\tilde{\lambda}_1$ .

First, we will study the behaviour of the flow in a neighborhood of the singularity and then use the information about the existence of a foliated atlas to obtain informations on the order of derivatives for the first return Poincaré maps.

Near the singularity, there exists a coordinate system such that the singularity  $p$  is in 0, the field is given by  $\tilde{X} = (\tilde{\lambda}_1 x, \tilde{\lambda}_2 y, \tilde{\lambda}_3 x)$  and there exists sections  $\tilde{\Sigma} = \{z = \varepsilon, |x| \leq \varepsilon, |y| \leq \varepsilon\}$ ,  $\tilde{\Sigma}_+ = \{x = +\varepsilon, \}$  and  $\tilde{\Sigma}_- = \{x = -\varepsilon\}$ .

By the same computations as for the Geometric Rovella  $X_0$  we have that the map from  $[-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon] \subset \tilde{\Sigma}$  to  $\tilde{\Sigma}_+$  is given by:

$$\tilde{F}(x, y, 1) = (1, \tilde{G}(x, y), \tilde{T}(x)) = (\varepsilon, \varepsilon^{-\tilde{r}} y \cdot x^{\tilde{r}}, \varepsilon^{1-\tilde{s}} x^{\tilde{s}}),$$

and that the time taken between  $\tilde{\Sigma}$  and  $\tilde{\Sigma}_+$  is given by

$$\tau(x, y, 1) = \frac{\log(\varepsilon) - \log(|x|)}{\tilde{\lambda}_1}.$$

Remark that both arguments work also for  $\tilde{\Sigma}_-$ .

We can choose the neighborhood such that  $\tilde{s} - 1 > 0$  and  $\tilde{r} - \tilde{s} > 3$ . Therefore, as  $x$  approaches 0 we have that

$$\frac{\partial \tilde{T}}{\partial x} = O(x^{\tilde{s}-1}), \quad \frac{\partial \tilde{G}}{\partial x} = O(x^{\tilde{r}-1}), \quad \frac{\partial \tilde{G}}{\partial y} = O(x^{\tilde{r}}),$$

with  $\tilde{s} > 1$ ,  $\tilde{r} > 1$ .

**8.2. Behaviour far from the fixed point.** If  $N \geq 3$  we know from [28] that all the vector fields in a neighborhood of  $X_0$  preserve a stable foliation. Let  $\tilde{X}$  be a vector field in a  $C^N$  neighborhood of  $X_0$  such that the resonance conditions are preserved, and whose flow preserves a stable foliation. Let  $F_{\tilde{X}}(x, y) = (T_{\tilde{X}}(x), G_{\tilde{X}}(x, y))$  be the first return map of the flow to the section  $\Sigma$ ; we will study the behaviour of its partial derivatives as  $x$  approaches 0.

With an abuse of notation, we will denote by  $\tilde{\Sigma}$ ,  $\tilde{\Sigma}_+$  and  $\tilde{\Sigma}_-$  the preimages under the linearizing change of coordinates of the sections  $\tilde{\Sigma}$ ,  $\tilde{\Sigma}_+$  and  $\tilde{\Sigma}_-$ ; the important property is that these preimages are locally  $C^1$ -manifolds and that they are made up of stable leaves. In particular, we can see  $\tilde{\Sigma}_+$  as a graph of a function of  $(y, z)$ , where the constant leaves are given by constant  $z$ .

We want to show that, since the flow preserves the stable foliation, then, recalling the results from subsection 8.1 we have that

- (1) the order of  $\frac{\partial \tilde{T}}{\partial x}$  as  $x$  goes to 0 is the same as the order of  $\frac{\partial \tilde{T}}{\partial x}$ , i.e.,  $\tilde{s} - 1$
- (2) the order of  $\frac{\partial \tilde{G}}{\partial y}$  is the same as the order of  $\frac{\partial \tilde{G}}{\partial y}$ , i.e.,  $\tilde{r}$
- (3) the order of  $\frac{\partial \tilde{G}}{\partial x}$  is at least the minimum of the orders of  $\frac{\partial \tilde{T}}{\partial x}$ ,  $\frac{\partial \tilde{G}}{\partial x}$ ,  $\frac{\partial \tilde{G}}{\partial y}$ , i.e., at least  $\tilde{s} - 1$ .
- (4) if  $\ln(x)$  is integrable with respect to the invariant measure, the return time to  $\Sigma$  is integrable.

Since the flow of  $\tilde{X}$  has no singularities besides  $p$ , the map  $\varphi_2(y, z) = (\mu(z), \nu(y, z))$  that takes  $\tilde{\Sigma}_+$  into  $\Sigma$  is a diffeomorphism that sends lines  $z = \text{const}$  into lines  $x = \text{const}$  and the map  $\varphi_1(x, y) = (\chi(x), \zeta(x, y))$  that takes  $\Sigma$  into  $\tilde{\Sigma}$  is a map that send lines  $x = \text{const}$  into lines  $x = \text{const}$ .

By direct computation we have that

$$\begin{aligned} DF_{\tilde{X}} &= D\varphi_2 \circ D\tilde{F} \circ D\varphi_1 \\ &= \begin{bmatrix} \frac{\partial \mu}{\partial z} \frac{\partial \tilde{T}}{\partial x} \frac{\partial \chi}{\partial x} & 0 \\ \frac{\partial \nu}{\partial y} \left( \frac{\partial \tilde{G}}{\partial x} \frac{\partial \chi}{\partial x} + \frac{\partial \tilde{G}}{\partial y} \frac{\partial \zeta}{\partial y} \right) + \frac{\partial \nu}{\partial z} \frac{\partial \tilde{T}}{\partial x} \frac{\partial \chi}{\partial x} & \frac{\partial \nu}{\partial y} \frac{\partial \tilde{G}}{\partial y} \frac{\partial \zeta}{\partial y} \end{bmatrix}. \end{aligned}$$

Since  $\varphi_2$  is a diffeomorphism, therefore  $\frac{\partial \nu}{\partial y} \cdot \frac{\partial \mu}{\partial z} \neq 0$ , and since  $\varphi_1$  is a diffeomorphism  $\frac{\partial \chi}{\partial x} \cdot \frac{\partial \zeta}{\partial y} \neq 0$ . This proves item (1),(2),(3).

Since the flow has no singularities both the time between  $\Sigma$  and  $\tilde{\Sigma}$  and the time between  $\tilde{\Sigma}_+$  and  $\Sigma$  are bounded; this implies item (4).

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